Structures and Intermittency in a Passive Scalar Model

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Perturbative expansions for intermittency scaling exponents in the Kraichnan passive scalar model [Phys. Rev. Lett. 72, 1016 (1994)] are investigated. A one-dimensional compressible model is considered for this purpose. High resolution Monte Carlo simulations using an Itô approach adapted to an advecting velocity field with a very short correlation time are performed and lead to clean scaling behavior for passive scalar structure functions. Perturbative predictions for the scaling exponents around the Gaussian limit of the model are derived as in the Kraichnan model. Their comparison with the simulations indicates that the scale-invariant perturbative scheme correctly captures the inertial range intermittency corrections associated with the intense localized structures observed in the dynamics. [S0031-9007(97)03871-4]

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Two major features of fully developed turbulent systems are the experimental evidence for scale invariance and intermittency in the inertial range of scales, and the presence of well-localized, coherent structures [1]. Geometrical properties of the latter depend on the specific cases, but their presence seems to be a generic feature of turbulent systems. For passive scalar transport, quasidiscontinuities of unit codimension have, in particular, been observed both numerically [2] and in the atmosphere [3]. The compatibility of structures with scale invariance and their dynamical relevance for intermittency in the inertial range have often been discussed in the literature (see [4] for a review), especially for 3D Navier-Stokes turbulence. No clear answer can be drawn, since numerical simulations have limited resolution and no \textit{ab initio} theory is available. The same type of questions arise for the class of white-in-time passive field models. The first and best known example is the passive scalar model introduced by Kraichnan [5]. Much interest has been attracted by this model since some understanding of its intermittency properties has been achieved. Equal-time correlation functions obey indeed closed equations of motion and anomalous scaling and intermittency are associated with zero modes of these equations [6–8]. Perturbative expansions around Gaussian limits of the model have been proposed to calculate the intermittency scaling exponents [6,7]. In both perturbative schemes, zero mode solutions are sought in a factorized scale-invariant form. Furthermore, structures of the kind observed in real turbulence are likely to be present also for the class of white-in-time passive field models. As a matter of fact, the anomaly in the nonperturbative solution for magnetic fields [9] comes from the balance between field lines stretching and eddy damping. This is indeed suggestive of the balancing mechanisms associated with structures. The issue of structures and intermittent scale invariance emerges then again, with the advantage that systematic questions can now be formulated. The one specifically addressed here is whether scale-invariant perturbative expansions correctly capture the scaling behavior in the inertial range and statistical effects of structures. Since the advecting velocity field considered in the Kraichnan model is far from the common experimental situations, the best candidates for this type of analysis are numerical simulations. Scaling laws in a 2D Kraichnan model are, however, hard to obtain with available resolutions [10]. The specific numerical difficulties of these simulations are related to the short correlation time of the random velocity field. We have therefore concentrated our attention on the one-dimensional scalar model presented here. As in the Kraichnan model, the velocity is Gaussian, \(\delta\) correlated in time, and its structure function scales with a positive exponent \(\xi\). The velocity is clearly compressible, but the model has a Gaussian limit for \(\xi \to 0\), and the perturbative expansion at small \(\xi\)’s can be performed exactly as for the incompressible Kraichnan model [7]. Structures are indeed observed in the form of strongly localized peaks, maintained by stretching due to velocity gradients. The comparison between theoretical predictions and numerical simulations is performed for the fourth- and the sixth-order structure functions at \(\xi = 0.5\).

The equation of the model is

\[ \partial_t \phi + \nabla (\nu \phi) = \kappa \Delta \phi + f. \]  

Here, both the velocity \(\nu\) and the injection \(f\) are Gaussian and \(\delta\) correlated in time. The velocity correlation function is

\[ \langle \nu(x,t)\nu(x',t') \rangle = \delta(t-t') [D_0 - S(|x-x'|)]. \]

The structure function \(S\) scales as \(S(x-x') = D|x-x'|^\xi\) (and \(0 < \xi < 2\)) in the range of scales between the ultraviolet and the infrared cutoffs \(\Lambda_{UV}\) and \(\Lambda_{IR}\) (the smallest and the largest scales in our problem). The large-scale injection satisfies \(\langle f(x,t)f(x',t') \rangle = \delta(t-t')F_L(x-x')\) with its Fourier transform \(\hat{F}_L\) concentrated near wave numbers \(O(1/L)\). It is assumed that \(\hat{F}_L\) vanishes at \(k = 0\), i.e., \(\int_{-\infty}^{\infty} F_L(x)dx = 0\). This condition is
associated with the fact that $\phi$ is a “gradient field.” The equation for the gradients in the Kraichnan model indeed coincides with (1), when vector components are omitted. The gradient nature of $\phi$ appears even more clearly from the solution for the second-order correlations presented later. The homogeneous part of the equation for the integrated field $\theta(x, t) = \int^{x} \phi(y, t) \, dy$ (scaling with positive exponents) is

$$\partial_{t} \theta + \nu \nabla \theta = \kappa \Delta \theta . \tag{3}$$

Note that the local maxima of $\theta$ cannot increase in time for (3), and the stretching process due to velocity gradients operates on the gradient field $\phi$. To fully resolve the structures, we have then preferred to deal directly with (1). This equation has been integrated by using a pseudospectral code with periodic boundary conditions. At each time step, a new realization of both $\nu$ and $f$ is generated. The multiplicative term in (1) is numerically treated in explicit form (à la Ito). The term differing in the formulations of Ito and Stratonovich [the latter being relevant for (1)] is taken into account [11].

The anomalous exponents are not expected to depend on the type of dissipation. For the measurement of scaling exponents, we have then used a $k^{8}$ hyperdissipation, but we have also checked that structures are present for normal dissipation. The resolution is $N = 2^{14}$. The injection is concentrated on the first mode. The velocity $\nu$ is simply generated in Fourier space.

A typical plot of the gradient field $\phi$ is shown in Fig. 1. The presence of very intense peaks of activity is evident. Their mechanism of formation (and persistence) can be immediately grasped from the original equation (1). Neglecting diffusion and forcing, the extrema $\tilde{\phi}$ of the field $\phi$ obey $\partial_{t} \tilde{\phi} = - ( \nabla \nu ) \tilde{\phi}$. Since $\nu$ is $\delta$ correlated in time, the logarithm of $\tilde{\phi}$ evolves as a Brownian motion. Furthermore, it is known that random walkers have a tendency not to change sign. Specifically, let the walk start from the origin. The cumulative distribution for the fraction of time spent on the negative side obeys the arc sine law [12]: Keeping the same sign for the whole walk and equipartition of the time between positive and negative values are the most and the least probable events, respectively. In our case, this implies that there will be very long periods of time when the stretching mechanism operates. Peaks such as those in Fig. 1 can then be maintained for significant times, despite the $\delta$ correlation of the velocity. This is indeed observed in our numerical simulations both for normal and hyperdissipation. Strong peaks persist long enough to make them clearly identifiable and meaningful; the use of the word “structures” to denote them. The structures are evidently the cause of the high tails of the probability distribution function (p.d.f.) in Fig. 2.

Let us now derive the predictions for the scaling exponents. The $\delta$ correlation in time ensures that correlation functions obey closed equations of motion. By using Gaussian integration by parts, it is, for example, easy to derive the equation for the second-order correlation $C_{2}(x, t) = \langle \phi(x, t) \phi(0, t) \rangle$:

$$\partial_{t} C_{2} = \frac{d^{2}}{dx^{2}} \left[ (2 \kappa + S) C_{2} \right] + F_L . \tag{4}$$

The solution is even and satisfies $\int C_{2} = 0$. In the inertial range of scales it decays with the negative

![FIG. 1. The gradient field $\phi$ vs the spatial coordinate $x$ for $\xi = 0.5$.](image1)

![FIG. 2. The probability distribution function (p.d.f.) of the gradient field $\phi$, normalized to its rms value $\sqrt{\langle \phi^2 \rangle}$.](image2)
exponent $-\xi$. The scaling of the second-order structure function of $\theta$ is obtained by simple integration. For $\xi < 1$, its inertial scaling is $2 - \xi$, as in the Kraichnan model.

Let us now consider the scaling of higher-order correlation functions. Closed equations of motion can be written for both the correlations $C_{2n} = \langle \phi(x_1) \cdots \phi(x_{2n}) \rangle$ and those of the integrated field $\theta$. Anomalous scaling is associated with zero modes of the homogeneous equations. The relevant equations for the zero mode $Z_{2n}$ of the $2n$-th order $\phi$ correlator are

$$\mathcal{H}Z_{2n} = \sum_{k+j,1}^{2n} \nabla_j \nabla_k [S(x_{jk})Z_{2n}] = 0. \quad (5)$$

Here, $x_{jk} = x_j - x_k$ and $\nabla_j$ stand for the spatial derivative with respect to $x_j$. Following [7], we look for a scale-invariant solution of this equation as an expansion in powers of $\xi$. Specifically, the structure functions are expanded as $S(x_{jk}) = D(1 + \xi \ln |x_{jk}|) + O(\xi^2)$ and $Z_{2n} = Z_{2n}^{(0)} + \xi Z_{2n}^{(1)} + O(\xi^2)$. The equations (for any $n$) at the lowest order in $\xi$ are trivially satisfied by taking $Z_{2n}^{(0)} = \text{const} = Z_{2n}$. At the first order in $\xi$, the solution is

$$Z_{2n}(x_1, \ldots, x_{2n}) = Z_{2n} \left[ 1 - \frac{\xi}{2} \sum_{k+j,1}^{2n} \ln |x_{jk}| \right]. \quad (6)$$

The scaling exponent (at order $\xi$) of the zero mode can now be obtained as in [13], operating on $Z_{2n}$ with the Euler operator $\mathcal{E} = \sum_{j=1}^{2n} \nabla_j$. It is easily checked that the equality $\mathcal{E}Z_{2n} = -\xi n(2n - 1)Z_{2n}$ holds at the first order in $\xi$. Since normal scaling would correspond to $-n\xi$, intermittency corrections are present, and their absolute value is $2n(n - 1)\xi$. This same value is obtained considering the equation for the correlations of the field $\theta$. The solution (6) at the first order in $\xi$ actually allows one to also obtain the corrections $O(\xi^2)$ to the scaling exponents. The procedure is the same as the one originally used in [7] at the first order in $\xi$ for the Kraichnan model. The equations at the second order are considered, and solvability conditions are imposed. The value of the correction $O(\xi^2)$ is thus obtained as the ratio of integrals that can be calculated numerically. The drawback of the procedure is that no check of the assumed self-similarity of the solution can be performed. The calculation has been carried out for the fourth order and gives a correction $9.389\xi^2$.

For comparison with numerical simulations, it is convenient to consider the scaling of structure functions $\langle |\theta(x, t) - \theta(0, t)|^p \rangle \sim |x|^{\xi_p}$. In particular, the prediction for the fourth and sixth order is

$$\xi_4 = 4 - 6\xi + O(\xi^2) \approx \frac{4}{1 + 1.5\xi}, \quad (7)$$

$$\xi_6 = 6 - 15\xi + O(\xi^2) \approx \frac{6}{1 + 2.5\xi},$$

where we have used the classical procedure of Padé approximants [14].

The $O(\xi^2)$ correction previously calculated for the fourth order can be used to construct two more nontrivial Padé approximants for $\xi_4$ and thus check the convergence of the procedure. For the comparison with numerical results, we have specifically considered $\xi = 0.5$. The measured second-order structure function is presented in Fig. 3. The fourth- and the sixth-order structure functions are shown in Fig. 4. The other two nontrivial Padé approximants for $\xi_4$ give $\sim 2.32$, indicating a reasonable robustness of the procedure for $\xi = 0.5$.

It is worth noting that the model presents nontrivial scaling also for low-order moments. The measured exponents for the moments of order $1/2, 1/8, -1/1, -0.3$ are $0.585, 0.163, -0.139,$ and $-0.444$. The corresponding normal values are $0.375, -0.094, -0.075,$ and $-0.25$. The $\xi_6$ curve then crosses the straight line of normal scaling at $p = 2$ and $p = 0$, is above it for $0 < p < 2$, and below it for negative moments (for orders smaller than $-1$, the moments do not exist). This behavior is qualitatively the same as in 3D Navier-Stokes turbulence (with one of the crossings occurring, however, at $p = 3$, and not $p = 2$, because of the 4/5 Kolmogorov law). Since the orders $p$ are low, strong peaks are likely to be irrelevant for this phenomenon. It is an interesting issue left for future work to identify the dynamical mechanisms leading to the observed low-order intermittency. Note also that $\xi = 1$ separates the cases of negatively ($\xi < 1$) and positively ($\xi > 1$) correlated increments of the velocity field. We are currently investigating the influence of positive correlations on structures and anomalous scaling.
FIG. 4. The measured fourth- (upper panel) and sixth- (lower panel) order structure functions for $\xi = 0.5$. Solid lines have the slopes predicted by (7), i.e., $\xi_4 = 2.29$ and $\xi_6 = 2.67$.

In conclusion, perturbative predictions for intermittency scaling exponents have been found to agree with numerical results. The comparison has been specifically performed in the one-dimensional case in order to have clean numerical scaling behaviors. However, the fact that the perturbative scheme is the same as the one originally proposed in [7] strongly suggests that the agreement carries over to the Kraichnan model. Note that, in the zero mode formalism, the structures and their dynamical properties do not appear explicitly, but only via their global statistical effects. Our results indicate that these effects are correctly taken into account by perturbative expansions around Gaussian limits. The agreement points in the direction of the scale-invariant zero mode mechanism for the intermittency of the Kraichnan passive scalar model.

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