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Point-source inertial particle dispersion

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The dispersion of inertial particles continuously emitted from a point source is analytically investigated in the limit of small but finite inertia. Our focus is on the evolution equation of the particle joint probability density function \( p(x, v, t) \), \( x \) and \( v \) being the particle position and velocity, respectively. For arbitrary inertia, position and velocity variables are coupled, with the result that \( p(x, v, t) \) can be determined by solving a partial differential equation in a 2\(d\)-dimensional space, \( d \) being the physical-space dimensionality. For small (but nevertheless finite) inertia, \((x,v)\)-variables decouple and the determination of \( p(x, v, t) \) is reduced to solve a system of two standard forced advection–diffusion equations in the space variables \( x \). The latter equations are derived here from first principles, i.e., from the well-known Lagrangian evolution equations for position and particle velocity.

**Keywords:** Multiphase and particle-laden flows; Mixing; Turbulent flows

1. Introduction

The study of particulate matter (PM) in flowing fluids is a problem attracting much attention because of its myriad of applications in different realms of science and technology. For example, we mention the implications on: climate dynamics and hydrological cycles (Lohmann and Feichter 2005, IPCC 2007), mainly in connection to global climate changes originated by PM-induced cloud formation (Celani et al. 2008); environmental sciences (Shi et al. 2001), in relation to pollution and deterioration of visibility; epidemiology (Dockery et al. 1993), in connection to adverse health effects in humans; and, finally, classical fluid dynamics, e.g., to understand how a flow field influences the particle concentration (Eaton and Fessler 1994, Balkovsky et al. 2001, Wilkinson and Mehlig 2003, Bec 2005, Bec et al. 2007).

The complex character of this discipline simply arises from the many interactions between totally different fields of science and technology. If, on the one hand, key
achievements have been obtained on both the epidemiological aspect and on the side of experimental studies, aiming at eliciting information on the distribution and number concentration in the environment, on the other hand very little is known about the dynamical aspect of particulate matter in situations of interest. As an instance, the basic equations ruling the spatio-temporal evolution of the particle concentration field are up to now unknown in the presence of paradigmatic environmental sources, such as, e.g., point- and line-source emissions (mimicking the release from a chimney and from a street, respectively). This seems to be the case in spite of the fact that the basic research dealing with inertial particles in fluids is a field in strong development, and where important results have been obtained in the last few years (see, e.g., Wilkinson and Mehlig 2003, Ruiz et al. 2004, Pavliotis and Stuart 2005).

Our specific aim here is to make a first step in the intermediate framework lying between the realm of abstract models of particle transport (sometimes too abstract to have direct implications on applicative fields, e.g., by neglecting particle inertia or gravity) and that of empirical models of transport (sometimes too empirical and with scarce contact with the underlying physics). Along this way, we will focus on the dynamics of PM when emitted from a point source. Note that only recently the point-source emission for the case of tracer particles (i.e., having the same density as the surrounding fluid and a very small size) has been addressed quantitatively from first principles (Celani et al. 2007). The main difference between the cases of tracers and inertial (i.e., massive and finite-size) particles is that, in the former case, the equations for the space/time evolution of particle concentration are very well known, and the resulting phenomenology has been elucidated only recently; in the latter case, we are still waiting for the governing field equations, which should be obtainable when inertia is small and the probability of caustic formation exponentially negligible. Here, our goal is to deduce such equations from first principles, i.e., from the known Lagrangian evolution of inertial particles (Gatignol 1983, Maxey and Riley 1983), exploiting well-founded methodologies from the realm of stochastic processes (see, e.g., Gardiner 1985, Risken 1989, Van Kampen 2007).

It is worth noticing that, in our approach, we take into account both particle and fluid (i.e., added mass) inertia, and both gravity and Brownian diffusivity. Actually, we will consider some of these effects in a simplified way, while others will automatically turn out to be subdominant. On the other hand, a significant role implicitly seems to be played by Brownian motion (the way we model the influence of the fluid thermal fluctuations on the particles), according to the nondimensionalization that we will operate and discuss (for an analysis of molecular dynamics and turbulent randomness, see, e.g., Russel (1981), Reeks (1988)). Other well-known models from the scientific literature, such as the “kinetic approach” (analog of the Maxwell–Boltzmann equation (Reeks 1991, 1992)) or the “mesoscopic approach” (moment equations derived from an integration of the particle density (see Simonin et al. 2002, Février et al. 2005 and references therein) look more suitable for specific applications and may be appropriate for any particle inertia. However, they need some ad hoc closure schemes (such as the Chapman–Enskog approximation for the third-order moment), or are applicable only for simple reference flows, e.g., steady/plane/parallel (see Smith 1988, 1990, and references therein) or shear/straining ones (Reeks 2005, Swailes et al. 2009). On the contrary, focusing only on small-inertia particles, we are able to deal with a generic incompressible fluid flow and to find results a priori.
This article is organized as follows. In section 2 we sketch the problem under consideration by recalling the significant equations and by performing quantitative balances to justify our approach. In section 3 we analyze our small-inertia expansion order by order, focusing on the related equations which stem out as solvability conditions. Conclusions follow in section 4. The appendix is devoted to briefly recalling the main steps leading to the Fokker–Planck equation for the phase-space density evolution, starting from the well-known Lagrangian equations ruling the particle evolution.

2. Statement of the problem

Let us consider the problem of dispersion of very dilute, small, spherical inertial particles emitted from a localized source, such as a chimney releasing some pollutant in the atmosphere or as an injection point (a syringe) in a microchannel, at a constant rate \( T/C_0 \). For the sake of simplicity, the spatial structure of the source is approximated as punctual, and located in the origin of our frame of reference (the case of distributed/instantaneous sources has been attacked in Smith (1988) and Swailes et al. (2009)). In terms of the release velocity, the emission distribution (denoted by \( f \)) will be left as unspecified (with the only constraint of normalization) for most of the general calculations, and then for illustrative applications will be modelled as a Gaussian, centered on an average value \( v/C_3 \) and with standard deviation \( \sigma/C_27 \). The axes are chosen such as to have \( v/C_3 \) aligned with the positive \( x_d \) direction (our polar axis): for the specific case of atmospheric pollution released from a smokestack, it vertically points upwards, i.e., opposed to the gravity acceleration \( g \), due to buoyancy effects. We consider as free parameters both the space dimension (even if the three-dimensional case should be kept in mind for applicative purposes) and the coefficient \( \beta/C_12 \), built from the particle \( \rho_p/C_26 \) and the fluid \( \rho_f/C_26 \) densities as follows:

\[
\beta = 3\rho_f/(\rho_f + 2\rho_p).
\]

We neglect some corrective terms (essentially due to Basset, Faxen, Oseen and Saffman (Maxey and Riley 1983)) but we take into account the added-mass effect in a simplified way, basically replacing the fluid acceleration with the corresponding velocity derivative computed along the particle trajectory, which amounts to introducing higher-order errors in the limit of small inertia (Bec 2003, Martins Afonso 2008). Therefore, the dynamical equations ruling the evolution of the particle position \( X(t) \) and covelocity \( V(t) \) in an incompressible flow \( u(x, t) \) read (Gatignol 1983; Maxey and Riley 1983)

\[
\dot{X}(t) = V(t) + \beta u(X(t), t), \quad (1a)
\]

\[
\dot{V}(t) = -\frac{V(t) - (1 - \beta)u[X(t), t]}{\tau} + (1 - \beta)g + \frac{\sqrt{2\kappa}}{\tau} \eta(t). \quad (1b)
\]

The Stokes time \( \tau \equiv R^2/(3v\beta) \) expresses the typical response delay of the particle to flow variations, i.e., the relaxation time in Stokes’ viscous drag \( R \) and \( v \) being the particle radius and the fluid viscosity, respectively; \( \eta(t) \) is the standard white noise associated to the particle Brownian diffusivity \( \kappa \), which was not considered in Gatignol (1983) and Maxey and Riley (1983). Notice that, when \( \beta \neq 0 \), there is a discrepancy between particle’s velocity, \( \dot{X}(t) \), and covelocity, \( \dot{V}(t) \equiv \dot{X}(t) - \beta u(X(t), t); \) for steady flows,
this means that, unless \( \rho_p \gg \rho_f \), the values of \( v_s \) and \( \sigma^2 \) should be interpreted as \( \bar{X}_{\text{emission}} - \beta u|_{x=0} \) and \( \bar{X}^2_{\text{emission}} + \beta^2 u^2|_{x=0} \), respectively (here the bar represents the average over the velocity distribution of the particle at the emission point).

We consider the external flow as unperturbed by the particles, which is a good approximation in our case of very dilute suspensions. Indeed, we remind that the fluid velocity \( u[X(t), t] \) in (1) must be the value found taking into account all of the disturbances due to boundaries and other particles (negligible in our situation) but excluding the particle under investigation itself. Of course, this does not rule out the possibility of having turbulent dispersion, besides Brownian dispersion and despite neglecting hydrodynamic interactions between particles. We can actually anticipate that we will end up with two forced passive-scalar equations, which are known to give rise to eddy diffusivities (Biferale et al. 1995, Mazzino 1997) when investigated at scales much larger than the ones typical of the advecting field.

If the flow has characteristic length scale and velocity \( L \) and \( U \), the forced Fokker–Planck equation for the particle phase-space density \( p(x, v, t) \) reads (Gardiner 1985, Risken 1989, Van Kampen 2007)

\[
\text{d}_t + \partial_{\mu}(v_{\mu} + \beta u_{\mu}) + \nabla_{\mu}\left[\frac{(1-\beta)u_{\mu} - v_{\mu} + (1-\beta)g_{\mu}}{\tau}\right] - \frac{\kappa}{\tau^2} \nabla^2 p = \frac{\delta(x)f(v)}{T},
\]

where \( \text{d}_t \equiv \partial/\partial t \), \( \partial_{\mu} \equiv \partial/\partial x_{\mu} \), and \( \nabla_{\mu} \equiv \partial/\partial v_{\mu} \). The derivation of this equation from (1) is a lengthy but straightforward task, and is briefly recalled in the appendix.

After performing the nondimensionalization (Martins Afonso 2008)

\[
\begin{align*}
&x \rightarrow \frac{x}{L}, \quad u \rightarrow \frac{u}{U}, \quad t \rightarrow \frac{t}{L/U}, \quad v \rightarrow \frac{v}{\sqrt{2\kappa/\tau}},
\end{align*}
\]

we can introduce the nondimensional numbers

- Stokes: \( \text{St} \equiv \frac{\tau U}{L} \),
- Froude: \( \text{Fr} \equiv \frac{U}{\sqrt{gL}} \),
- Pécelt: \( \text{Pe} \equiv \frac{LU}{\kappa} \),

and the vertical unit vector pointing downwards: \( \mathbf{G} \equiv \mathbf{g}/g \).

Note that the particle covelocity \( v \) has a nondimensionalization different from the fluid velocity \( u \). Let us then try to explain the meaning of this nondimensionalization with \( \sqrt{2\kappa/\tau} \), in view of our investigation in the limit of small inertia. In the phase space, this can be seen as resulting from a competition between the small-\( \tau \) limit, which would reduce the phase-space dimension by making \( v \) collapse onto the variety \( u(x, t) \), and the effect of diffusivity, which counteracts this process. In order to have the nondimensionalized covelocity \( \sim O(1) \) (i.e., a meaningful nondimensionalization) and, at the same time, the dimensional one \( \sim O(U) \) (because of the small-inertia limit, which is usually associated with small deviations of the particle from the underlying fluid trajectory),

\[\text{This “passive” point of view, corresponding to a one-way coupling, actually has a huge range of applicability, namely for mass loadings sufficiently smaller than unity (see, e.g., Li et al. 2001). To mention just one example, if we consider an EU PM10 standard concentration of } 40 \mu g m^{-3} \text{ in the atmospheric boundary layer, the mass loading is trivially } 4 \times 10^{-8} \text{, therefore one can safely neglect any two-way coupling up to particulate concentrations of almost } 10^7 \text{ times the threshold fixed by the European Community.}\]
the following is a sufficient condition:
\[
\frac{\sqrt{2\kappa}}{r} \sim \mathcal{O}(v) \sim U \implies \frac{2}{\text{St Pe}} \sim \mathcal{O}(1) \implies (loosely) \text{St Pe} \sim \mathcal{O}(1).
\] (3)

3. Expansion at small inertia

In terms of the nondimensional variables introduced in the previous section, (2) becomes
\[
\left\{-\text{St}^{-1}\left(\nabla v - \frac{1}{2} \nabla^2\right) + \text{St}^{-1/2}\left[\frac{2}{\text{Pe}} v \partial_\mu + \sqrt{\frac{\text{Pe}}{2}} (1 - \beta) u_\mu \nabla_\mu\right]
\right.
\]
\[
\left.\quad + \text{St}^0\left[d_i + \beta u_\mu \partial_\mu\right] + \text{St}^{1/2}\left[\sqrt{\frac{\text{Pe}}{2}} (1 - \beta) G_\mu \nabla_\mu\right]\right\} u = \frac{L}{T U} \delta(x) f(v).\] (4)

Let us now perform a small-Stokes expansion of the particle probability density function (PDF) and a Hermitianization of the problem in the spirit of Martins Afonso (2008):
\[
p(x, v, t) = \sum_{n=0}^{\infty} \text{St}^{n/2} p_n(x, v, t), \quad p_n(x, v, t) = e^{-v^2/2} \psi_n(x, v, t).\] (5)

Notice that, because \(p\) is normalized to unity, each \(p_n\) must be normalized to \(\delta_{n0}\).

Substituting (5) into (4), we obtain a chain of equations for \(\psi_n\) at the different orders in \(\text{St}\), which we analyze in detail in what follows.

3.1. Order \(\text{St}^{-1}\)

We have
\[
(\nabla^2 - v^2 + d) \psi_0 = 0 \implies \psi_0 = e^{-v^2/2} \xi_0(x, t),\] (6)
where \(\xi_0\) is still unknown at this stage (we only know that it must be normalized to \(\pi^{-d/2} = (\int dv e^{-v^2})^{-1}\)), and will be found by imposing the solvability condition at \(\mathcal{O}(\text{St}^0)\).

3.2. Order \(\text{St}^{-1/2}\)

We have
\[
(\nabla^2 - v^2 + d) \psi_1 = e^{-v^2/2} v_\mu \left[2\sqrt{2}\text{Pe}^{-1/2} \partial_\mu - 2\sqrt{2} (1 - \beta) \text{Pe}^{1/2} u_\mu\right] \xi_0
\]
\[
\implies \psi_1 = e^{-v^2/2} \left[\xi_1(x, t) + v_\mu \rho_{1\mu}(x, t)\right]
\]
\[
\implies \rho_{1\mu} = \sqrt{2} (1 - \beta) \text{Pe}^{1/2} u_\mu \xi_0 - \sqrt{2} \text{Pe}^{-1/2} \partial_\mu \xi_0,
\] (7)
where \(\xi_1\) is still unknown at this stage (we only know that it must be normalized to 0), and will be found by imposing the solvability condition at \(\mathcal{O}(\text{St}^{1/2})\).
3.3. Order St⁰

We have

\[(\nabla^2 - v^2 + d)\psi_2\]

\[= -2 \frac{L}{TU} \delta(x) f(v)e^{v^2/2} + 2e^{-v^2/2} \left\{ \left[ d_t + (2\beta - 1)u_\mu \partial_\mu + (1 - \beta)^2 Pe u^2 \right] \xi_0 + v_\mu v_\nu \left[ 2(1 - \beta)(\partial_\mu u_\nu) + 4(1 - \beta)u_\mu \partial_\nu - 2Pe^{-1}\partial_\mu \partial_\nu - 2(1 - \beta)^2 Pe u_\mu u_\nu \right] \xi_0 + v_\mu \left[ \sqrt{2}Pe^{-1/2}\partial_\mu - \sqrt{2}(1 - \beta)Pe^{1/2}u_\mu \right] \xi_1 \right\} .\]  

(8)

At this point, we have to impose the solvability condition (which would have been trivial at the lower orders (6), (7)):

\[\int \! dv \, e^{-v^2/2/(\nabla^2 - v^2 + d)} \psi_2 = \int \! dv \, \psi_2 (\nabla^2 - v^2 + d) e^{-v^2/2} = 0 \]

\[\implies 0 = -2 \frac{L}{TU} \delta(x) \int \! dv \, f(v) + 2 \left[ d_t + (2\beta - 1)u_\mu \partial_\mu + (1 - \beta)^2 Pe u^2 \right] \xi_0 \int \! dv \, e^{-v^2}
+ 2 \left[ 2(1 - \beta)(\partial_\mu u_\nu) + 4(1 - \beta)u_\mu \partial_\nu - 2Pe^{-1}\partial_\mu \partial_\nu - 2(1 - \beta)^2 Pe u_\mu u_\nu \right] \xi_0 \int \! dv \, v_\nu v_\nu e^{-v^2}
+ 2 \left[ \sqrt{2}Pe^{-1/2}\partial_\mu - \sqrt{2}(1 - \beta)Pe^{1/2}u_\mu \right] \xi_1 \int \! dv \, v_\mu e^{-v^2},\]

and thus

\[(d_t + u \cdot \nabla - Pe^{-1}\nabla)\xi_0 = \frac{L}{TU} \pi^{d/2} \delta(x).\]  

(9)

This means that, in the limit St → 0, the nondimensional PDF is

\[p \rightarrow p_0 = e^{-v^2/2} \psi_0 = e^{-v^2} \xi_0,\]

where \(\xi_0\) behaves as a passive scalar with spatial point source; moreover, after integrating on the covelocity variable to obtain the physical-space density,

\[P(x, t) \equiv \int \! dv \, p(x, v, t),\]  

(10)

in dimensional quantities one obtains the advection–diffusion equation for a passive-scalar field (investigated from first principles in Celani et al. (2007) for a point-source emission):

\[(d_t + u \cdot \nabla - \kappa \partial^2)P = \frac{1}{T} \delta(x) \quad \text{(for St → 0)}\]

(the derivation of a “gradient–diffusion”, or Smoluchowski, equation from the generalized Fokker–Planck was discussed in Chandrasekhar (1943)).
Now, substituting \(d_t \xi_0\) from (9) back to (8), one gets

\[
(\nabla^2 - v^2 + d) \psi_2 = -2 \frac{L}{TU} \delta(x) \left[ f(v)e^{v^2/2} - \pi^{-d/2}e^{-v^2/2} \right]
+ 2e^{-v^2/2} \left[ \left( -2(1 - \beta)u_\mu \partial_\mu + Pe^{-1} \partial_\mu^2 + (1 - \beta)^2 Pe u_\mu^2 \right) \xi_0 \right.
+ v_\mu v_i \left( 2(1 - \beta)(\partial_\mu u_\nu) + 4(1 - \beta)u_\mu \partial_\nu - 2 Pe^{-1} \partial_\mu \partial_\nu - 2(1 - \beta)^2 Pe u_\mu u_\nu \right) \xi_0
+ v_\mu \left[ \sqrt{2}Pe^{-1/2} \partial_\mu - \sqrt{2}(1 - \beta)Pe^{-1/2} u_\mu \right] \xi_1 \bigg],
\]

and thus

\[
\psi_2 = e^{-v^2/2} \left[ \xi_2(x, t) + v_\mu \rho_2^\mu(x, t) + v_\mu v_i \sigma_2^{\mu i}(x, t) \right] - 2 \frac{L}{TU} \delta(x) \phi(v),
\]

where \(\xi_2, \rho_2, \sigma_2\) can be derived by extending the investigation at higher orders in St (not reported here), and \(\phi\) satisfies the forced equation

\[
(\nabla^2 - v^2 + d) \phi = f(v)e^{v^2/2} - \pi^{-d/2}e^{-v^2/2}.
\]

### 3.4. Order \(\text{St}^{1/2}\)

We have

\[
(\nabla^2 - v^2 + d) \psi_3
= -2\sqrt{2} \frac{L}{TU} \left[ (1 - \beta)Pe^{1/2} u_\mu \delta(x)(\nabla_\mu - v_\mu) \phi(v) + 2 Pe^{-1/2} v_\mu \phi(v) \partial_\mu \delta(x) \right]
+ 2e^{-v^2/2} \left[ \left( d_t + (2\beta - 1)u_\mu \partial_\mu + (1 - \beta)^2 Pe u_\mu^2 \right) \xi_1 \right.
+ v_\mu v_i \left( 2(1 - \beta)(\partial_\mu u_\nu) + 4(1 - \beta)u_\mu \partial_\nu - 2 Pe^{-1} \partial_\mu \partial_\nu - 2(1 - \beta)^2 Pe u_\mu u_\nu \right) \xi_1
+ v_\mu \left[ (\ldots)\xi_0 + (\ldots)\xi_2 \right] + v_\mu v_i v_\lambda (\ldots)\xi_0 \bigg].
\]

The solvability condition has to be imposed again at this point:

\[
\int dv e^{-v^2/2}(\nabla^2 - v^2 + d) \psi_3 = \int dv \psi_3(\nabla^2 - v^2 + d)e^{-v^2/2} = 0
\]

\[
\Rightarrow 0 = -2\sqrt{2} \frac{L}{TU} \int dv e^{-v^2/2} \left[ (1 - \beta)Pe^{1/2} u_\mu \delta(x)(\nabla_\mu - v_\mu) \phi(v) + 2 Pe^{-1/2} v_\mu \phi(v) \partial_\mu \delta(x) \right]
+ 2 \left[ d_t + (2\beta - 1)u_\mu \partial_\mu + (1 - \beta)^2 Pe u_\mu^2 \right] \xi_1 \int dv e^{-v^2}
+ 2 \left[ 2(1 - \beta)(\partial_\mu u_\nu) + 4(1 - \beta)u_\mu \partial_\nu - 2 Pe^{-1} \partial_\mu \partial_\nu - 2(1 - \beta)^2 Pe u_\mu u_\nu \right] \xi_1 \int dv v_\mu v_i e^{-v^2}
+ 2 [(\ldots)\xi_0 + (\ldots)\xi_2] \int dv v_\mu e^{-v^2} + 2(\ldots)\xi_0 \int dv v_\mu v_i v_\lambda e^{-v^2},
\]
and thus
\[
(d_t + u \cdot \partial - Pe^{-1} \partial^2)\xi_t = 2\sqrt{2} \frac{L}{TU} \pi^{-d/2} Pe^{-1/2} [\partial_{\mu} \delta(x)] \int dv \ e^{-r^2/2} v_{\mu} \phi(v). \tag{13}
\]

Let us consider two different cases, corresponding (I) to the presence and (II) to the absence of rotational symmetry in \( f. \)

(I) In this case \( f = f(v), \) which means a centered and isotropic emission covelocity, corresponding to a source emitting particles with velocity \( \beta n(0, t) \) plus isotropic fluctuations (meaningful for \( \beta = 0 \) if \( v_\ast = 0 \)). If this is the case, then from (12) also \( \phi = \phi(v) \) and the integral on the right-hand side (RHS) of (13) vanishes because of parity, therefore \( \xi_t = \text{const.} = 0 \) and (from (7))

\[
\psi_1 = e^{-r^2/2} v_{\mu} \left[ \sqrt{2(1 - \beta)} Pe^{1/2} u_{\mu} - \sqrt{2} Pe^{-1/2} \partial_{\mu} \right] \xi_0.
\]

When projecting onto the physical space through (10), this term integrates to zero, and no explicit correction appears with respect to the tracer approximation:

\[
P = e^{-r^2} \xi_0 + O(\text{St}).
\]

(II) On the contrary, if \( f(v) \) has a preferential direction (a mean value \( v_\ast \) along the positive \( x_d \) coordinate) one has to decompose \( \phi = \sum_{l,d,m} \phi(l,m) Y_{l,m} \) onto the spherical harmonics and obtains a quantum-harmonic-oscillator-like equation. As \( f \) shows degeneracy on the azimuthal angle, i.e., no preferential direction on the plane orthogonal to the mean emission, the same happens for \( \phi, \) which thus reduces to a sum of Legendre polynomials (spherical harmonics with \( m = 0 \)). With the substitution into the RHS of (13) in mind, one can discard the isotropic sector and, if, e.g.,

\[
f(v) = (2\pi\sigma^2)^{-d/2} e^{-(v-v_\ast)^2/2\sigma^2},
\]

from (12) one gets

\[
[l_l^2 + (d - 1)v^{-1} \partial_v - l(l + d - 2)v^{-2} - v^2 + d] \phi_l(v) = e^{r^2/2} i \sqrt{4\pi(l + 1)} (2\pi\sigma^2)^{-d/2} e^{-r^2/2\sigma^2 - v^2/2\sigma^2} J_l(-2iv\nu/\sigma^2)
\]

(with \( l \geq 1 \), \( J_l \) being the spherical Bessel function of the first kind. Thus,

\[
\phi_l(v) = v^l e^{-r^2/2} \left\{ \mathcal{U} \left( \frac{l}{2}, \frac{d}{2} + l, v^2 \right) \left[ C_{l} - K \int_0^v dk k^{l+d-1} \mathcal{L}_{l/2}^{(d/2+l-1)}(k^2) f_l(k) \right] 
+ \mathcal{L}_{l/2}^{(d/2+l-1)}(v^2) \left[ c_{l} + K \int_0^\infty dk k^{l+d-1} \mathcal{U} \left( \frac{l}{2}, \frac{d}{2} + l, k^2 \right) f_l(k) \right] \right\}, \tag{14}
\]

where the values of \( C_l \) and \( c_l \) can be found by imposing regularity and normalization, and the constant \( K \) is given by \( \forall k \)

\[
K \equiv \frac{k^{-d-2} \sqrt{\pi} \mathcal{U} \left( 1 + l/2, 1 + d/2 + l, k^2 \right) \mathcal{L}_{l/2}^{(d/2+l-1)}(k^2) - 2 \mathcal{U} \left( l/2, d/2 + l, k^2 \right) \mathcal{L}_{l/2-l/2}^{(d/2+l-1)}(k^2)}{2\mathcal{U} \left( l/2, d/2 + l, k^2 \right) \mathcal{L}_{l/2-l/2}^{(d/2+l-1)}(k^2)}.
\]
(whose value for \(l=1\) is \(\pi/2\) in \(d=3\), and \(\sqrt{\pi}\) in \(d=2\)). Here, \(\mathcal{L}\) is the generalized Laguerre function and \(\mathcal{U}\) is Kummer’s confluent hypergeometric function (of the second kind) (Gradshteyn and Ryzhik 1965).

One can further focus on the first anisotropic sector \(l=1\) (the only non-vanishing when substituting in (13), as easily provable thanks to the orthogonality of spherical harmonics), which implies \(C_{(1)} = 0 = c_{(1)}\) for regularity. As a result, the RHS of (13) reduces to

\[
4 \sqrt{\frac{2}{3}} \pi^{-(d-1)/2} \frac{L}{TU} \text{Pe}^{-1/2} [\delta_{\chi, \phi}] \int dv v^d e^{-v^2/2} \phi_{(1)}(v),
\]

the latter integral being just a number.

4. Conclusions

Gathering all of our information, for the particle PDF we have

\[
p(x, v, t) = e^{-v^2/2} [\psi_0(x, v, t) + \text{St}^{1/2} \psi_1(x, v, t)] + \text{O(St)}
\]

\[
= e^{-v^2} \xi_0(x, t) + \text{St}^{1/2} e^{-v^2} \left\{ \xi_1(x, t) + v \mu \left[ \sqrt{2} (1 - \beta) \text{Pe}^{1/2} u_\mu(x, t) \xi_0(x, t) - \sqrt{2} \text{Pe}^{-1/2} \partial_\mu \xi_0(x, t) \right] \right\} + \text{O(St)},
\]

(15)

with \(\xi_0\) and \(\xi_1\) obeying the following forced advection–diffusion equations:

\[
(d_t + u \cdot \partial - \text{Pe}^{-1} \delta^2) \xi_0 = \frac{L}{TU} \pi^{-d/2} \delta(x),
\]

\[
(d_t + u \cdot \partial - \text{Pe}^{-1} \delta^2) \xi_1 = 4 \sqrt{\frac{2}{3}} \pi^{-(d-1)/2} \frac{L}{TU} \text{Pe}^{-1/2} [\delta_{\chi, \phi}] \int dv v^d e^{-v^2/2} \phi_{(1)}(v)
\]

(and \(\phi_{(1)}(v)\) expressed by (14) and following hints).

It is worth stressing that the original Eulerian description of inertial particle concentration involves equations in a \(2d\)-dimensional phase space (\(d\) for the space coordinates plus \(d\) for the particle velocity coordinates). This fact makes it prohibitive to numerically solve such a system. Our approach dramatically reduces the degrees of freedom to the sole space coordinates: rather than a partial differential equation in a \(2d\) space, one has to solve two innocent forced advection–diffusion passive-scalar equations in physical space. This is a rather trivial numerical exercise.

From (15), one can obtain information on different physical properties, or otherwise deduce the physical-space PDF (10) directly:

\[
P(x, t) = \pi^{d/2} \xi_0(x, t) + \text{St}^{1/2} \pi^{d/2} \xi_1(x, t) + \text{O(St)}.
\]

Notice that, up to the order we have investigated (\(\text{O(St}^{1/2}\)), gravity plays no role. In other words, particle sedimentation, which is known to take place with a (bare) terminal

\[\text{\ldots}\]
velocity $\propto \text{St}$ plus flow-induced corrections, thus gives rise to a subleading effect. If one wanted to have a non-negligible gravitational effect already at $O(\text{St}^{1/2})$, one should take the limits $\text{St} \to 0 \leftarrow \text{Fr}$ at the same time (i.e., small inertia coupled with strong gravity or large free-fall velocity), in order to keep $\text{StFr}^{-2}$ finite and thus to get a description of sedimentation at the leading-correction order.

In order to give a practical example of application of our equations, let us concentrate on micro-powder dispersion in microchannels. In a typical microchannel size of $L \sim 10^{-4}$ m, inside which distilled water at temperature $T \sim 320 \text{K}$ (density $\rho_t \sim 988$ kg m$^{-3}$ and kinematic viscosity $\nu \sim 5 \times 10^{-7}$ m$^2$ s$^{-1}$) is flowing at the typical velocity $U \sim 10^{-4}$ m s$^{-1}$, we inject lead micro-powder containing particles of size $R \sim 2 \times 10^{-6}$ m having a density $\rho_p \sim 11.3 \times 10^3$ kg m$^{-3}$. The Stokes time is $\tau \sim 2 \times 10^{-5}$ s and the particle diffusivity is $\kappa \sim k_B T / (6\pi \nu R) \sim 2.4 \times 10^{-13}$ m$^2$ s$^{-1}$ (from Einstein’s relation, $k_B = 1.38 \times 10^{-23}$ J K$^{-1}$ being Boltzmann’s constant). In this way, the Péclet number is $\text{Pe} \equiv LU/\kappa \sim 4 \times 10^4$, the Stokes number is $\text{St} \equiv \tau U / L \sim 2 \times 10^{-5}$, and thus $\text{StPe} = O(1)$, consistently with (3). In other words, this case represents an example in which the thermally-induced fluctuations in the particle velocity,

$$v_k \equiv \sqrt{\frac{2k_B T}{m_p}} \sim \sqrt{\frac{9 \nu \rho_p \kappa}{\rho_p R^2}} \sim \sqrt{\frac{2 \kappa}{\tau}}$$

($m_p$ being the particle mass), are such that $v_k \sim O(U)$. Note that, despite the fact that St is quite small, the first correction in our equation does appear at $O(\text{St}^{1/2})$, thus giving a more appreciable $5 \times 10^{-3}$ which perfectly falls in the present perturbative scheme. Similar relevant examples that we could mention concern the deposition of powders in filters or in the alveoli regions in lungs.

In all the previous instances, $\kappa$ has to be interpreted as the Brownian diffusivity. However, following a point of view already considered, e.g., in Swailes et al. (2009), we can also extend our approach to deal with problems of particulate dispersion in the atmospheric boundary layer after its emission from a chimney. Let us indeed suppose to now separate the velocity field into a resolved part plus a nonresolved (i.e., parameterized) one, as is customary in the Reynolds-Averaged Navier–Stokes (RANS) approach. In this case, $\kappa$ is rather to be identified with the eddy diffusivity. A reasonable estimation for this coefficient is $10^{-1}$ m$^2$ s$^{-1}$, reported, e.g., in the seminal paper by Sutton (1932). For example, we focus on a PM10 particle with radius $R \sim 10^{-5}$ m and density $\rho_p \sim 2 \times 10^3$ kg m$^{-3}$, the air density and viscosity being $\rho_t \sim 1$ kg m$^{-3}$ and $\nu \sim 10^{-5}$ m$^2$ s$^{-1}$, respectively. Assuming the deterministic part of the velocity field to be characterized by $L \sim 200$ m and $U \sim 5$ m s$^{-1}$, we get a Stokes number of $\text{St} \sim 1.1 \times 10^{-4}$ and a Péclet number of $\text{Pe} \sim 10^4$, fitting our constraint (3). The leading correction to the particle concentration is thus $O(\text{St}^{1/2})$, i.e., order percent.

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References


Appendix A: The Fokker–Planck equation associated to the Lagrangian description

The derivation of the Fokker–Planck equation (2) from the Lagrangian equations (1) is briefly recalled here by exploiting the corresponding unforced (sourceless) case (see, e.g., Klyatskin 2005).

The distribution \( p(x, v, t) \) is given by the average (on the realizations of the random noise \( \eta \)) of the product of two Dirac deltas, expressing the probability density that, at time \( t \), the particle location \( X \) and covelocity \( V \) equal the spatial coordinate \( x \) and the covelocity variable \( v \), respectively:

\[
p(x, v, t) \equiv \langle \delta(x - X(t))\delta(v - V(t)) \rangle \quad (A.1)
\]

(notice that, in this expression and in the following ones, the average acts on the random functions \( X \) and \( V \), which are \( \eta \)-dependent, and not on \( x \) and \( v \)). A differentiation in time and the use of the chain rule give

\[
\frac{\partial p}{\partial t} = \left\{ \frac{\partial}{\partial t} \delta(x - X(t)) \right\} \delta(v - V(t)) + \delta(x - X(t)) \frac{\partial v}{\partial V(t)} \frac{\partial}{\partial v} \delta(v - V(t))
\]

Exploiting (1) and the translational invariance of \( \delta \), but keeping the term explicitly depending on \( \eta \) momentarily apart, we get

\[
\frac{\partial p}{\partial t} = \left( \frac{V(t) + \beta u(X(t), t)}{\tau} - \frac{\partial}{\partial x} \right) \delta(v - V(t)) + \delta(x - X(t)) \frac{\partial v}{\partial V(t)} \frac{\partial}{\partial v} \delta(v - V(t))
\]

\[
\times \left\{ \frac{V(t) - (1 - \beta)u[X(t), t]}{\tau} + (1 - \beta)g \right\} \left[ - \frac{\partial}{\partial v} \delta(v - V(t)) \right] + \langle \eta(t) \ldots \rangle.
\]

The derivatives can now be moved out of the brackets and, inside the curly braces, the capital letters can be replaced by the corresponding small ones because of the deltas:

\[
\frac{\partial p}{\partial t} = - \frac{\partial}{\partial x} \left\{ (v_\mu + \beta u_\mu(x, t)) \delta(x - X(t))\delta(v - V(t)) \right\}
\]

\[
- \frac{\partial}{\partial v_\mu} \left\{ - \frac{v_\mu - (1 - \beta)u_\mu(x, t)}{\tau} + (1 - \beta)g_\mu \right\} \delta(x - X(t))\delta(v - V(t))
\]

\[
+ \langle \eta(t) \ldots \rangle.
\]

Moving the curly braces out of the averages, applying (A.1), and moving to the left-hand side the first two terms on the RHS, we are left with

\[
\left[ \frac{\partial}{\partial t} + \frac{\partial}{\partial x_\mu} (v_\mu + \beta u_\mu(x, t)) + \frac{\partial}{\partial v_\mu} \left( \frac{(1 - \beta)u_\mu(x, t) - v_\mu}{\tau} + (1 - \beta)g_\mu \right) \right] p
\]

\[
= \left\{ \delta(x - X(t)) \frac{\sqrt{2\kappa}}{\tau} \eta_\mu(t) \frac{\partial}{\partial V_\mu(t)} \delta(v - V(t)) \right\} \equiv [\star].
\]

(A.2)
If one takes into account the source contribution, which forces the addition of a term $\delta(x)f(v)/T$ on the RHS of (A.2) (this can be interpreted as the imposition of a boundary condition), the derivation of (2) is thus complete once the stochastic term $[\nu]$ is made explicit, which can easily be done by means of a Gaussian integration by parts (here, $D$ represents the functional derivative) (Frisch 1995):

$$\begin{align*}
[\nu] &= \frac{\sqrt{2\kappa}}{\tau} \int_{-\infty}^{\infty} \eta_{\mu}(t') \left\{ \frac{D}{D_{\eta_{\mu}}(t')} \left[ -\delta(x - X(t)) \frac{\partial \delta(v - V(t))}{\partial v_{\mu}} \right] \right\} \\
&= -\frac{\sqrt{2\kappa}}{\tau} \frac{\partial}{\partial v_{\mu}} \int_{-\infty}^{\infty} \eta_{\mu}(t') \left\{ \frac{D}{D_{\eta_{\mu}}(t')} \left[ \delta(x - X(t)) \delta(v - V(t)) \right] \right\} \\
&= -\frac{\sqrt{2\kappa}}{\tau} \frac{\partial}{\partial v_{\mu}} \left\{ \frac{D}{D_{\eta_{\mu}}(t')} \left[ \delta(x - X(t)) \delta(v - V(t)) \right] \right\}.
\end{align*}$$

Integrating (1) formally in time and invoking causality (a variation of $\eta$ at time $t$ cannot cause any change in $X$ or $V$ at the same instant, but only at later times), in the last expression the former functional derivative vanishes and, in the latter, only the term explicitly showing $\eta$ survives:

$$\begin{align*}
[\nu] &= -\frac{\sqrt{2\kappa}}{\tau} \frac{\partial}{\partial v_{\mu}} \left\{ \frac{D}{D_{\eta_{\mu}}(t')} \left[ \delta(x - X(t)) \delta(v - V(t)) \right] \right\} \\
&= \frac{2\kappa}{\tau^2} \frac{\partial^2}{\partial v_{\mu} \partial v_{\lambda}} \left\{ \delta(x - X(t)) \delta(v - V(t)) \right\} \int_{-\infty}^{\infty} \eta_{\mu}(t') \left\{ \frac{D}{D_{\eta_{\mu}}(t')} \left[ \delta(x - X(t)) \delta(v - V(t)) \right] \right\} \\
&= \frac{\kappa}{\tau^2} \frac{\partial^2}{\partial v_{\mu} \partial v_{\lambda}} \delta(t - t'').
\end{align*}$$

(the Heaviside theta is indeed such that $\theta(0) = 1/2$).