ONE DIMENSIONAL WAVE PROPAGATION

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What is a wave?

1. A wave is a disturbance $\phi$ that propagates in a medium.

2. $\phi$ can be a displacement, a velocity, a stress, or a pressure.

3. The medium can be a solid or a fluid.

4. In one dimension, the disturbance is a function of the position $x$ and time $t$ so $\phi(x, t)$.

5. When $t=0$, the function $f(X) = \phi(x, 0)$ represents an instantaneous photograph of the disturbance at that time also called the **wave profile**.
Harmonic wave

\[ \phi(x, t) = a \cos k(x - ct) \]

- Cosine function has a period equal to \(2\pi\)
- At a given instant, the wavelength \(\lambda\) is such that \(k\lambda = 2\pi\)
- At a given location, the period \(T\) is such that \(2\pi = kcT\)

Wave number \(k = \frac{2\pi}{\lambda}\)  
Frequency: \(\omega = \frac{2\pi}{T} = kc\)
Harmonic waves

Harmonic waves are usually sinusoidal functions of the form

\[ \phi(x, t) = \cos k(x - ct) \quad \text{or} \quad \phi(x, t) = \sin k(x - ct) \]

Why study this type of waves:

- A pulse consists of many frequencies (Fourier transform)
- When a pulse cannot propagate freely we consider the propagation of individual components
- Each component may propagate at a different speed (dispersion)
- Sometimes the wave propagates but is attenuated (amplitude decays)
- Sometimes the wave cannot propagate at all (evanescent waves)
At a particular instant, the wavelength is the period in space

\[ \phi(x, 0) \]

\[ \lambda = \frac{2\pi}{k} \]

\[ \phi(x, t) = a \sin k(x - ct) \]

\[ k \lambda = 2\pi \]

\[ k = \frac{2\pi}{\lambda} \text{ wave number} \]
At a particular point, $T$ is the period in time

$$\phi(0, t)$$

$$\phi(x, t) = \sin k(x - ct)$$

$$kcT = 2\pi$$

$$kc = \frac{2\pi}{T} = \omega$$

circular frequency: $\omega = kc$

Substituting $\phi(x, t) = \sin k(x - ct)$ into the equations of motion (to be determined) will give a relationship between $k$ and $\omega$ or $k$ and $c$: the dispersion relation
What is a rod?

- A slender body: length $L$ is much longer than the size of the cross section
- Subjected to tension
- Uniaxial stress
- A shaft has the same basic geometry but it is subjected to a torque
- A beam has the same basic geometry but it is subjected to transverse loads
Equation of motion for a rod (elementary approach)

\[ F \rightarrow \text{rod} \rightarrow F + dF \]

Newton’s law

\[ \rho A \Delta x \frac{\partial^2 u}{\partial t^2} = dF \]

\[ dF = \frac{\partial F}{\partial x} \cdot dx = \frac{\partial (A \sigma)}{\partial x} \cdot dx \]

At this point nothing has been assumed about:

1. The cross-sectional area: it could be constant or it could vary with \( x \).
2. The stress-strain behavior of the material: linear or non-linear, constant or varying with \( x \).
Linear elastic rod with uniform cross section

\[\rho A \frac{\partial^2 u}{\partial t^2} = \frac{\partial (A\sigma)}{\partial x}\]

\[\rho \frac{\partial^2 u}{\partial t^2} = \frac{\partial \sigma}{\partial x}\]

Hooke’s law

\[\sigma = E \varepsilon\]

\[\rho \frac{\partial^2 u}{\partial t^2} = E \frac{\partial \varepsilon}{\partial x}\]

Linear strain-displacement relation

\[\varepsilon = \frac{\partial u}{\partial x}\]

\[\rho \frac{\partial^2 u}{\partial t^2} = E \frac{\partial^2 u}{\partial x^2}\]

\[\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}\]

Classical form of wave equation
Free wave propagation in uniform linear elastic rod

Wave equation for the rod
\[ \rho \frac{\partial^2 u}{\partial t^2} = E \frac{\partial^2 u}{\partial x^2} \]

Seek solution of the form:
\[ u = f(\xi) \quad \text{with} \quad \xi = x - ct \quad (\text{arbitrary shaped pulse propagating from left to right}) \]

\[ \frac{\partial u}{\partial x} = \frac{\partial f}{\partial \xi} \frac{\partial f}{\partial x} = \frac{\partial f}{\partial \xi} \]

\[ \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 f}{\partial \xi^2} \]

\[ \frac{\partial u}{\partial t} = c \frac{\partial f}{\partial \xi} \]

\[ \frac{\partial^2 u}{\partial x^2} = c^2 \frac{\partial^2 f}{\partial \xi^2} \]

A pulse of arbitrary shape will propagate with a constant velocity
\[ c = \sqrt{\frac{E}{\rho}} \]
\[ c = \begin{cases} 
\text{Wave velocity} \\
\text{Phase velocity} \\
\text{Acoustic wave speed} \\
\text{Speed of sound in the material} 
\end{cases} \]
Harmonic waves in a uniform linear elastic rod

\[
\frac{\partial^2 u}{\partial t^2} = \frac{E}{\rho} \frac{\partial^2 u}{\partial x^2}
\]

\[
u = a \cos k(x - ct) = a \cos(kx - \omega t)
\]

\[
-\omega^2 a \cos(kx - \omega t) = -\frac{E}{\rho} k^2 a \cos(kx - \omega t)
\]

\[
\omega^2 = \frac{E}{\rho} k^2
\]

\[
c^2 = \frac{\omega^2}{k^2} = \frac{E}{\rho}
\]

- All harmonic waves propagate with the same speed \( c = \sqrt{E/\rho} \)
- For a pulse consisting of several frequencies, all components travel at the same speed so the pulse does not change shape as it propagates
- The wave equation (this model for the rod) is non-dispersive.
Displacement, velocity, strain, and stress are all governed by the wave equation

\[ c^2 \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2} \]

Differentiate with respect to time \( c^2 \frac{\partial^2}{\partial x^2} \left( \frac{\partial u}{\partial t} \right) = \frac{\partial^2}{\partial t^2} \left( \frac{\partial u}{\partial t} \right) \quad \Rightarrow \quad c^2 \frac{\partial^2 v}{\partial x^2} = \frac{\partial^2 v}{\partial t^2} \)

Differentiate with respect to x \( c^2 \frac{\partial^2}{\partial x^2} \left( \frac{\partial u}{\partial x} \right) = \frac{\partial^2}{\partial t^2} \left( \frac{\partial u}{\partial x} \right) \quad \Rightarrow \quad c^2 \frac{\partial^2 \varepsilon}{\partial x^2} = \frac{\partial^2 \varepsilon}{\partial t^2} \)

Multiply by \( E \)

\[ \frac{\partial^2 (E \varepsilon)}{\partial x^2} = \frac{1}{c^2} \frac{\partial^2 (E \varepsilon)}{\partial t^2} \quad \Rightarrow \quad c^2 \frac{\partial^2 \sigma}{\partial x^2} = \frac{\partial^2 \sigma}{\partial t^2} \]

Important because in some problems any one of those quantities can be prescribed.
Other problems governed by the wave equation

**Uniaxial strain** (flyer plate experiment):

Dimensions in the y and z directions are much larger than the thickness (x-direction)

\[
\begin{align*}
\varepsilon_{yy} = \varepsilon_{zz} &= 0 \\
\begin{bmatrix}
\varepsilon_{xx} \\
\varepsilon_{yy} \\
\varepsilon_{zz}
\end{bmatrix} &= \frac{1}{E} \begin{bmatrix}
1 & -\nu & -\nu \\
-\nu & 1 & -\nu \\
-\nu & -\nu & 1
\end{bmatrix}
\begin{bmatrix}
\sigma_{xx} \\
\sigma_{yy} \\
\sigma_{zz}
\end{bmatrix}
\end{align*}
\]

\[
\sigma_{xx} = E^* \varepsilon_{xx}
\]

Effective modulus

\[
E^* = \frac{E(1-\nu)}{(1+\nu)(1-2\nu)}
\]

for isotropic materials

\[
c = \sqrt{E^* / \rho}
\]
Transverse vibration of a string

Small deformations:

- Transverse displacement \( v \) is small
- Rotation angle \( \frac{\partial v}{\partial x} \) is small

The tension \( T \) remains constant.

\( m \) is the mass per unit length

\[
\frac{\partial^2 v}{\partial t^2} = c^2 \frac{\partial^2 v}{\partial x^2}
\]

with \( c = \sqrt{T/m} \)
Short history of the vibrating string

- Brook Taylor (Taylor series expansion): *Methodus Incrementorum* (1715)
- Johann Bernoulli (1728) used Leibniz’s notation and found the mode shapes for a vibrating string. No wave equation
- D’Alembert (1747) derived the equation of motion for a string (wave equation) and found that the general solution is the sum of two waves: one propagating towards the right and one propagating towards the left.
- Euler (1748) found solution to the initial value problem in terms of two waves
- Daniel Bernoulli (1753): vibration of finite string, modal superposition, no way to find the amplitudes
- Lagrange (1759) modal superposition
Torsion of a shaft

\[ \frac{\partial^2 \theta}{\partial t^2} = c^2 \frac{\partial^2 \theta}{\partial x^2} \]

\[ c = \sqrt{G/\rho} \]
**Exponential rod**

Equation of motion for an elastic rod with varying cross section

\[ \rho A \frac{\partial^2 u}{\partial t^2} = \frac{\partial (A \sigma)}{\partial x} \]

\[ \rho A \frac{\partial^2 u}{\partial t^2} = \frac{\partial}{\partial x} \left( AE \frac{\partial u}{\partial x} \right) \]

\[ \rho A \frac{\partial^2 u}{\partial t^2} = AE \frac{\partial^2 u}{\partial x^2} + E \frac{\partial A}{\partial x} \frac{\partial u}{\partial x} \]

Cross section varies exponentially

\[ A = A_0 e^{2\alpha x} \]

\[ \rho \frac{\partial^2 u}{\partial t^2} = E \frac{\partial^2 u}{\partial x^2} + 2\alpha E \frac{\partial u}{\partial x} \]

For harmonic waves \( u = f(x)e^{i\omega t} \)

\[ E \frac{d^2 f}{dx^2} + 2\alpha E \frac{df}{dx} + \rho \omega^2 f = 0 \]
\[ f = e^{\beta x} \rightarrow E\beta^2 + 2\alpha E\beta + \rho \omega^2 = 0 \rightarrow c_o^2\beta^2 + 2\alpha c_o^2 \beta + \omega^2 = 0 \]

\[ \beta = -2\alpha \pm i \left( \sqrt{\frac{\omega^2}{c_o^2} - \alpha^2} \right) = -2\alpha \pm i\kappa \rightarrow f = e^{-2\alpha x} \left\{ A_1 e^{i\kappa x} + A_2 e^{-i\kappa x} \right\} \]

\[ u = e^{-2\alpha x} \left\{ A_1 e^{i(\kappa x + \omega t)} + A_2 e^{-i(\kappa x - \omega t)} \right\} \]

1. First term: exponential decay due to change in cross section (no damping)
2. Terms in parentheses: propagating waves
3. Wave number depends on frequency: exponential rod is dispersive
String with distributed damping

\[ T \frac{\partial^2 v}{\partial x^2} - C \frac{\partial v}{\partial t} = m \frac{\partial^2 v}{\partial x^2} \]

\[ v = Ae^{i(kx-\omega t)} \]

\[ Tk^2 = m \omega^2 + i\omega C \]

\[ k = k_R + ik_I \]

\[ v = Ae^{-k_I x} e^{i(k_R x - \omega t)} \]

The wave propagates but the amplitude decays exponentially because of damping
String with elastic foundation

Equation of motion \[ T \frac{\partial^2 v}{\partial x^2} - K v = m \frac{\partial^2 v}{\partial x^2} \]

\[ v = A e^{i(kx-\omega t)} \]

Dispersion relation \[ T k^2 + K = m \omega^2 \]

\[ \omega^2 = \frac{T}{m} k^2 + \frac{K}{m} \]

\[ \omega_{\text{cutoff}} = \sqrt{K / m} \]

What happens if \( \omega < \omega_{\text{cutoff}} ? \)

\[ k^2 = \left( m \omega^2 - K \right) / T \]

If \( m \omega^2 > K \), \( k \) is real, propagating waves

If \( m \omega^2 < K \), \( k \) is imaginary \( k = i k' \), evanescent waves

\[ v = A e^{-k' x} e^{-\omega t} \]
Lamb waves (harmonic waves in an elastic layer)

- Horace Lamb (1917)
- Equations of motion from 2D elasticity, stress-free conditions on top and bottom surfaces
- Symmetric and anti-symmetric harmonic waves
- $S_0$ wave is extensional wave initially. Phase velocity is constant in elementary theory
- $A_0$ is bending wave initially. In Bernoulli-Euler beam theory $\omega = c^2 \sqrt{\rho A/El}$ (parabola)
- At higher frequencies, the phase velocities for $A_0$ and $S_0$ tend to $c_R$, the velocity of Rayleigh waves
D’Alembert’s solution to the wave equation

\[
\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}
\]

1. Change of variables to transform equation of motion
2. Solution in terms of two propagating waves
3. Initial value problems
4. Boundary value problems
D’Alembert’s solution

\[ \frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} \]

Change of variables: \( x, t \rightarrow \zeta, \eta \) to be determined

\[
\frac{\partial^2 u}{\partial x^2} = \left( \frac{\partial^2 u}{\partial \zeta^2} \frac{\partial \zeta}{\partial x} + \frac{\partial^2 u}{\partial \zeta \partial \eta} \frac{\partial \eta}{\partial x} \right) \frac{\partial \zeta}{\partial x} + \left( \frac{\partial u}{\partial \eta} \frac{\partial \zeta}{\partial x} + \frac{\partial^2 u}{\partial \eta^2} \frac{\partial \eta}{\partial x} \right) \frac{\partial \eta}{\partial x}
\]

\[
\frac{\partial^2 u}{\partial t^2} = \left( \frac{\partial^2 u}{\partial \zeta^2} \frac{\partial \zeta}{\partial t} + \frac{\partial^2 u}{\partial \zeta \partial \eta} \frac{\partial \eta}{\partial t} \right) \frac{\partial \zeta}{\partial t} + \left( \frac{\partial u}{\partial \eta} \frac{\partial \zeta}{\partial t} + \frac{\partial^2 u}{\partial \eta^2} \frac{\partial \eta}{\partial t} \right) \frac{\partial \eta}{\partial t}
\]

Substitute into the wave equation
\[
\begin{align*}
c^2 \left( \frac{\partial^2 u}{\partial \xi^2} + \frac{\partial^2 u}{\partial \xi \partial \eta} \right) \frac{\partial \xi}{\partial x} + c^2 \left( \frac{\partial u}{\partial \eta \partial \xi} \frac{\partial \xi}{\partial x} + \frac{\partial^2 u}{\partial \xi^2} \frac{\partial \eta}{\partial x} \right) \frac{\partial \eta}{\partial x} &= \\
\left( \frac{\partial^2 u}{\partial \xi^2} \frac{\partial \xi}{\partial t} + \frac{\partial^2 u}{\partial \xi \partial \eta} \frac{\partial \xi}{\partial t} \right) \frac{\partial \xi}{\partial t} + \left( \frac{\partial u}{\partial \eta \partial \xi} \frac{\partial \xi}{\partial t} + \frac{\partial^2 u}{\partial \xi^2} \frac{\partial \eta}{\partial t} \right) \frac{\partial \eta}{\partial t} &
\end{align*}
\]

Three types of terms: \( \frac{\partial^2 u}{\partial \xi^2} \), \( \frac{\partial^2 u}{\partial \eta^2} \), \( \frac{\partial^2 u}{\partial \xi \partial \eta} \). To eliminate the first two

\[
\frac{\partial^2 u}{\partial \xi^2} \cdot c^2 \left( \frac{\partial \xi}{\partial x} \right)^2 = \left( \frac{\partial \xi}{\partial t} \right)^2 \quad \frac{\partial^2 u}{\partial \eta^2} \cdot c^2 \left( \frac{\partial \eta}{\partial x} \right)^2 = \left( \frac{\partial \eta}{\partial t} \right)^2
\]

These are requirements for the new coordinates. The wave equation becomes

\[
c^2 \frac{\partial^2 u}{\partial \xi \partial \eta} \frac{\partial \xi \partial \eta}{\partial x \partial x} = \frac{\partial^2 u}{\partial \xi \partial \eta} \frac{\partial \xi \partial \eta}{\partial t \partial t}
\]
Using
\[ c^2 \left( \frac{\partial \zeta}{\partial x} \right)^2 = \left( \frac{\partial \zeta}{\partial t} \right)^2 \quad \rightarrow \quad c \frac{\partial \zeta}{\partial x} = \frac{\partial \zeta}{\partial t} \]
\[ c^2 \left( \frac{\partial \eta}{\partial x} \right)^2 = \left( \frac{\partial \eta}{\partial t} \right)^2 \quad \rightarrow \quad c \frac{\partial \eta}{\partial x} = -\frac{\partial \eta}{\partial t} \]

we obtain the “canonical form” of the wave equation

\[ \frac{\partial^2 u}{\partial \zeta \partial \eta} = 0 \]

Integrating with respect to \( \eta \) first:
\[ \frac{\partial u}{\partial \zeta} = f(\zeta) \quad \rightarrow \quad u = F(\zeta) \]

Integrating with respect to \( \zeta \) first:
\[ \frac{\partial u}{\partial \eta} = g(\eta) \quad \rightarrow \quad u = G(\eta) \]

General solution
\[ u = F(\zeta) + G(\eta) \]
To go back to the original variables $x$, $t$

$$c \frac{\partial \zeta}{\partial x} = \frac{\partial \zeta}{\partial t} \quad \rightarrow \quad \zeta = x + ct$$

$$c \frac{\partial \eta}{\partial x} = -\frac{\partial \eta}{\partial t} \quad \rightarrow \quad \eta = x - ct$$

D’Alembert’s solution

$$u = F(x + ct) + G(x - ct)$$

Propagates to the left

Propagates to the right
Initial value problem

\[
\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}
\]

with initial conditions

\[
\begin{align*}
\ u(x,0) &= f(x) \\
\ \frac{\partial u}{\partial t}(x,0) &= g(x)
\end{align*}
\]

General solution \( u = F(x + ct) + G(x - ct) \)

Initial conditions

\[
\begin{align*}
\ u(x,0) &= f(x) = F(x) + G(x) \\
\ \frac{\partial u}{\partial t}(x,0) &= g(x) = cF'(x) - cG'(x)
\end{align*}
\]

Integrating from \( a \) to \( x \), \( 2^{\text{nd}} \) IC gives

\[
F(x) - G(x) = F(a) - G(a) + \frac{1}{c} \int_a^x g(s)ds
\]

\[
F(x) = \frac{1}{2} \left\{ F(a) - G(a) + f(x) + \frac{1}{c} \int_a^x g(s)ds \right\} \\
G(x) = \frac{1}{2} \left\{ -F(a) + G(a) + f(x) - \frac{1}{c} \int_a^x g(s)ds \right\}
\]

\[
u(x, t) = \frac{1}{2} \left\{ f(x - ct) + f(x + ct) + \frac{1}{c} \int_{x-ct}^{x+ct} g(s)ds \right\}
\]
Saint Venant’s solution

\[ u(x, t) = \frac{1}{2} \left\{ f(x - ct) + f(x + ct) + \frac{1}{c} \int_{x-ct}^{x+ct} g(s) ds \right\} \]

- initial displacement
- initial velocity

(x, t)

\[ x - ct = \text{constant} \quad x + ct = \text{constant} \]

\[ \text{Domain of dependence of } u(x,t) \]
Example 1

\[ u(x,0) = \begin{cases} 
1 & \text{for } 0 < x < 1 \\
0 & \text{otherwise} 
\end{cases} \quad \dot{u}(x,0) = 0 \]

\[ u(x, t) = \frac{1}{2} \{ f(x - ct) + f(x + ct) \} \]
Initial displacements
Boundary value problems

- Pressure applied on a half-space
- Reflection at a free end
- Reflection at a fixed end
- Incident, reflected, and transmitted waves at an interface
Pressure applied on half-space

Solution: one wave propagating towards the right \( \sigma(x, t) = f(x - ct) \)

On a characteristic line \( x - ct = \) constant, the stress is constant (whatever is applied at \( x=0 \))

\[
T(x, t) = -\sigma(x, t) = p \left( t - \frac{x}{c} \right)
\]
Mechanical impedance

\[ u(x, t) = f(x - ct) \]

\[ \varepsilon = \frac{\partial u}{\partial x} = f'(x - ct) \]

\[ v = \frac{\partial u}{\partial t} = -cf'(x - ct) \]

In general

\[ \sigma = E\varepsilon = -E \frac{v}{c} = -\frac{E}{\rho} v = -\rho cv \]

\[ T = -\sigma = \rho cv = zv \]

Mechanical Impedance
Number the regions separated by characteristic lines

Region 0: at rest, $T=0$, $v=0$

Region 2: $T=p$, $v=p/Z$

Region 3: at rest, $T=0$, $v=0$

Stress at one point as a function of time
Reflection at free surface

\[ T_i, v_i = \frac{T_i}{z} \]

\[ T = 0, v = 0 \]

\[ T_R = -T_i \]

\[ v_R = -\frac{T_R}{z} = \frac{T_i}{z} = v_i \]

\[ T = 0, v = 0 \]

\[ T = T_i, v = \frac{T_i}{z} \]

\[ T = 0, v = 2v_i \]
Reflection at fixed end

- $T = 2T_1, \quad v = 0$
- $T = T_i, \quad v = T_i / z$
- $T = 0, \quad v = 0$
For bars with different cross sections

\[ \sigma_T = \sigma_i \frac{2\rho_2 c_2 A_2}{\rho_1 c_1 A_1 + \rho_2 c_2 A_2} \]

\[ \sigma_R = \sigma_i \frac{\rho_2 c_2 A_2 - \rho_1 c_1 A_1}{\rho_1 c_1 A_1 + \rho_2 c_2 A_2} \]

• When \( z_2 > z_1 \) reflected and incident waves have same sign
• When \( z_2 < z_1 \) reflected and incident waves have opposite sign
• Resultant stress on bar 1 has same sign as incident stress
Impact of elastic rod against rigid wall

\[ T = zv_o \]

Duration = \( 2L/c \)

\( L = 1 \text{ m}, c = 5000 \text{ m/s}, t = 400 \mu \text{s} \)
Hugoniot experiment
Pulse on elastoplastic rod

\[ \sigma_0 = \sigma_Y \]

\[ \rho = \frac{1}{E} \frac{d\sigma}{d\varepsilon} \]

\[ c_E = \sqrt{\frac{E}{\rho}} \]

\[ c_P = \sqrt{\frac{E_1}{\rho}} \]
Stress at a particular point as a function of time

State of stress along the bar at a particular instant

Stress at a particular point as a function of time
Mono-atomic spring-mass chain

\[
u_{i+1} \quad m \quad k \quad m \quad k \quad m
\]

\[\hat{\kappa} = \kappa_R + i \kappa_i\]

Substituting into equation of motion gives two equations:

\[-m \omega^2 + 2k [1 - \cos(\kappa_R a)cosh(\kappa_i a)] = 0\]
\[\sin(\kappa_R a)\sinh(\kappa_i a) = 0\]

\[\sinh(\kappa_i a) = 0 \quad \omega = \pm 2\sqrt{k/m} \sin(\kappa_R a / 2)\]
\[\sin(\kappa_R a) = 0 \quad \cosh(\kappa_i a) = \frac{m}{2k} \omega^2 - 1\]

\[u = U \exp[\pm i(\kappa_R x - \omega t)]\]
\[u = U \exp(-\kappa_i x)\exp[-i\omega t]\]
Dispersion curve for linear spring-mass chain

- Just one pass band
- Beyond cut-off frequency waves cannot propagate
- Different frequencies propagate at different speeds. The chain is dispersive
- Linear spring-mass chain acts as a low-pass mechanical filter

\[ \omega_0 = \sqrt{\frac{k}{m}} \]
Steady state response of a 20 mass uniform chain with a harmonic force applied on mass 1 and fixed at the other end.
Diatomic Chain

\[
m_1 \ddot{u}_n + k_1 (u_n - v_n) + k_2 (u_n - v_{n-1}) = 0
\]
\[
m_2 \ddot{v}_n + k_1 (v_n - u_n) + k_2 (v_n - u_{n+1}) = 0
\]
\[
u = U e^{i(\kappa x - \omega t)} \quad v = V e^{i(\kappa x - \omega t)}
\]

\[
m_1 m_2 \omega^4 - (m_1 + m_2)(k_1 + k_2)\omega^2 + 4k_1 k_2 \sin^2(\kappa a / 2) = 0
\]

Dispersion curves \((k_1 = k_2 = 1, m_1 = 1, m_2 = 2)\)
Wave propagation in chain with branches

\[ m_1 \ddot{u}_n + k_1 (2u_n - u_{n-1} - u_{n+1}) + k_2 (u_n - v_n) = 0 \]
\[ m_2 \ddot{v}_n + k_2 (v_n - u_n) = 0 \]

\[ \omega^4 m_1 m_2 - \omega^2 \left\{ 4 \sin^2 \left( \frac{\kappa a}{2} \right) k_1 m_2 + k_2 (m_1 + m_2) \right\} + 4 \sin^2 \left( \frac{\kappa a}{2} \right) k_1 k_2 = 0 \]

Example: \( m_1=1, k_1=1, m_2=0.1 \) and \( k_2=0.1 \)

Adding the branches created a bandgap
Continuum modeling of discrete chain

\[ m \ddot{u}_n + k (2u_n - u_{n-1} - u_{n+1}) = 0 \]

\[ u = u_n + \frac{\partial u}{\partial x} \Delta x + \frac{1}{2} \frac{\partial^2 u}{\partial x^2} (\Delta x)^2 + \frac{1}{6} \frac{\partial^3 u}{\partial x^3} (\Delta x)^3 + \frac{1}{24} \frac{\partial^4 u}{\partial x^4} (\Delta x)^4 + \frac{1}{120} \frac{\partial^5 u}{\partial x^5} (\Delta x)^5 + \frac{1}{720} \frac{\partial^6 u}{\partial x^6} (\Delta x)^6 \]

\[ u_{n+1} = u_n + \frac{\partial u}{\partial x} a + \frac{1}{2} \frac{\partial^2 u}{\partial x^2} a^2 + \frac{1}{6} \frac{\partial^3 u}{\partial x^3} a^3 + \frac{1}{24} \frac{\partial^4 u}{\partial x^4} a^4 + \frac{1}{120} \frac{\partial^5 u}{\partial x^5} a^5 + \frac{1}{720} \frac{\partial^6 u}{\partial x^6} a^6 \]

\[ u_{n-1} = u_n - \frac{\partial u}{\partial x} a + \frac{1}{2} \frac{\partial^2 u}{\partial x^2} a^2 - \frac{1}{6} \frac{\partial^3 u}{\partial x^3} a^3 + \frac{1}{24} \frac{\partial^4 u}{\partial x^4} a^4 - \frac{1}{120} \frac{\partial^5 u}{\partial x^5} a^5 + \frac{1}{720} \frac{\partial^6 u}{\partial x^6} a^6 \]

\[ u_{n+1} + u_{n-1} = 2u_n + \frac{\partial^2 u}{\partial x^2} a^2 + \frac{1}{12} \frac{\partial^4 u}{\partial x^4} a^4 + \frac{1}{360} \frac{\partial^6 u}{\partial x^6} a^6 \]

\[ \frac{\partial^2 u}{\partial t^2} = \frac{ka^2}{m} \left( \frac{\partial^2 u}{\partial x^2} + \frac{1}{12} \frac{\partial^4 u}{\partial x^4} a^2 + \frac{1}{360} \frac{\partial^6 u}{\partial x^6} a^4 \right) \]
\begin{align*}
  m \ddot{u}_n + k \left(2u_n - u_{n-1} - u_{n+1}\right) &= 0 \\
  \omega^2 &= \frac{2k}{m} \left(1 - \cos(\kappa a)\right) \\
  u &= U \exp[i(\kappa x - \omega t)] \\
  \cos x &\approx 1 - \frac{x^2}{2} + \frac{x^4}{24} - \frac{x^6}{7202} \\
  \omega^2 &= \frac{2k}{m} \left\{ \frac{(\kappa a)^2}{2} - \frac{(\kappa a)^4}{24} + \frac{(\kappa a)^6}{720} \right\} \\
  \frac{\partial^2 u}{\partial t^2} - \omega^2 &= -\kappa^2 \\
  \frac{\partial^2 u}{\partial x^2} - \kappa^2 &= \partial^4 u - \kappa^4 \\
  \frac{\partial^6 u}{\partial x^6} - \kappa^6 &= \\
  -\frac{\partial^2 u}{\partial t^2} &= \frac{2k}{m} \left\{ -\frac{a^2}{2} \frac{\partial^2 u}{\partial x^2} - \frac{a^4}{24} \frac{\partial^4 u}{\partial x^4} - \frac{a^6}{720} \frac{\partial^6 u}{\partial x^6} \right\} \\
  \frac{\partial^2 u}{\partial t^2} &= \frac{ka^2}{m} \left\{ \frac{\partial^2 u}{\partial x^2} + \frac{a^2}{12} \frac{\partial^4 u}{\partial x^4} + \frac{a^4}{360} \frac{\partial^6 u}{\partial x^6} \right\}
\end{align*}
Continuum models for elastic chains

\[ \frac{\partial^2 u}{\partial t^2} = \frac{ka^2}{m} \frac{\partial^2 u}{\partial x^2} \]

\[ \frac{\partial^2 u}{\partial t^2} = \frac{ka^2}{m} \left\{ \frac{\partial^2 u}{\partial x^2} + \frac{a^2}{12} \frac{\partial^4 u}{\partial x^4} + \frac{a^4}{360} \frac{\partial^6 u}{\partial x^6} \right\} \]

\[ \frac{\partial^2 u}{\partial t^2} = \frac{ka^2}{m} \left\{ \frac{\partial^2 u}{\partial x^2} + \frac{a^2}{12} \frac{\partial^4 u}{\partial x^4} \right\} \]

\[ ka = 2\pi \frac{a}{\lambda} \]